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A note on complete sets of material conservation laws

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An attempt is made to derive complete sets of conservation laws for various flows. It is shown that the equations of three-dimensional adiabatic flow cannot be transformed into a complete set of conservation laws. It is demonstrated that potential vorticity is the only material invariant of shallow-water flow. A complete set can be derived for three-dimensional homogeneous flow, if Lagrangian tracers are added.

1. Introduction

Conservation laws are of central importance to fluid dynamics. For example, the discovery of the conservation of potential vorticity by Ertel (1942) led to a wealth of new insights in meteorology and oceanography (see Hoskins, McIntyre & Robertson 1985, and Bryan 1987 for reviews). Ertel's discovery initiated a search for additional conservation laws (e.g. Ertel & Rossby 1949; Fortak 1956; Hollmann 1964). In general one may look at a flow in N-dimensional space represented by K variables r_k and governed by M diagnostic and $K-M$ prognostic equations. A set of conservation laws

$$\frac{\mathrm{d}c_i}{\mathrm{d}t} = 0 \quad (i = 1, \ldots, I) \tag{1.1}$$

is complete if the flow evolution can be determined in time through (1.1) if all material invariants $c_i(\boldsymbol{x}, t)$ are given at $t = 0$, say. This means, in particular, that the velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ can be determined from $c_i(\boldsymbol{x}, t)$:

$$\boldsymbol{v} = \mathscr{V}(c_i), \tag{1.2}$$

where the symbol $\mathscr{V}$ denotes an operator. If, for example, $\boldsymbol{v}$ must be derived from vorticity q of two-dimensional homogeneous flow one has to invert the Laplacian in $\nabla^2\psi = q$ to obtain the stream function ψ. With $\boldsymbol{v} = \boldsymbol{k} \times \nabla\psi$ one has $\mathscr{V} = (\boldsymbol{k} \times \nabla)\nabla^{-2}$ (see §3.2). In general boundary conditions are needed as well to determine $\boldsymbol{v}$. In what follows we shall impose periodic boundary conditions to avoid complications. Note that (1.2) does not contain time explicitly. Of course, similar relationships hold for other flow variables like density or temperature if a complete set of invariants exists. We require in turn that all c_i can be determined if the flow variables are known: $c_i = c_i(r_k)$. The functional relation $c_i(r_k)$ may also involve differentiation of the variables (see, for example, (2.5)–(2.7)). However, the definition of an invariant should not involve time explicitly. In many cases it proves convenient to add new variables and corresponding equations to the original flow description and to search then for a complete set of material invariants for the extended problem. For example, Hollmann (1964) added an 'action function' W (Wirkungsfunction; see (2.4)) to the basic variables of adiabatic three-dimensional flow in order to obtain a complete set of conservation laws. However, Dikyi (1972) pointed out that Hollmann's set is not complete since W cannot be derived from the invariants (see also Mobbs 1981). So we are still left with the problem if there exists a complete set of conservation laws for an adiabatic fluid. More generally one may search for a complete set for any type of flow where, of course, only inviscid unforced flows are admitted. For example, a complete set is known for two-dimensional homogeneous flow (see §3) but has not yet been derived for three-dimensional homogeneous flow. At this point it is convenient to introduce Lagrangian tracers $X_j, j = 1-N$, where the position vector $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) = (X_1, \ldots, X_N)$ gives the position of a particle in space at time $t = 0$ which is located at $\boldsymbol{x}$ at time t, so that $\boldsymbol{X}(\boldsymbol{x}, 0) = \boldsymbol{x}$. Here and in what follows we assume that the flow is sufficiently smooth so that the relation $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$ can be inverted to yield $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$. The Lagrangian tracers are conserved

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = 0. \tag{1.3}$$

Now let us look at a flow and assume that a complete set of I material invariants exists. The flow equations may include (1.3). If not we add (1.3) to the system. The extended set must be complete, too. We can integrate the assumed complete set of conservation laws (1.1) in time (except (1.3), of course) to obtain I relations

$$c_i(\boldsymbol{x}(\boldsymbol{X}, t), t) = c_i(\boldsymbol{X}, 0). \tag{1.4}$$

The extended system contains then $M+I$ diagnostic relations and the N prognostic equations (1.3). This means that only N prognostic equations of first order are needed to integrate a flow problem in time where a complete set of invariants exists. We could have arrived at the same result by introducing N out of all c_i as Lagrangian tracers (provided $I \geqslant N$).

2. Three-dimensional adiabatic flow

The equations of three-dimensional adiabatic flow read

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - \nabla\phi, \tag{2.1}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}, \tag{2.2}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0, \tag{2.3}$$

where the components of velocity $\boldsymbol{v}$, density ρ and potential temperature θ are the variables, $\phi = gz$ is geopotential and p is pressure with $p = (\theta\rho R/p_0^k)^{1/1-k}$, $k = R/c_p$, p_0 a constant reference pressure.

Obviously $c_1 = \theta$ is conserved. Ertel (1942) has shown that potential vorticity $c_2 = \nabla\theta\cdot\boldsymbol{q}/\rho$ is also conserved where $\boldsymbol{q} = \nabla\times\boldsymbol{v}$ is vorticity.

Up to now no additional invariant has been found for (2.1)–(2.3). Hollmann extended (2.1)–(2.3) by adding the action function W as a variable which is predicted according to

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \tfrac{1}{2}v^2 - \phi - c_p T, \tag{2.4}$$

where $T = \theta(p/p_0)^\kappa$ and $W = 0$ at $t = 0$. Hollmann showed that

$$c_3 = \frac{1}{\rho}(\nabla c_1 \times \nabla c_2)\cdot \boldsymbol{A}, \tag{2.5}$$

$$c_4 = \frac{1}{\rho}(\nabla c_1 \times \nabla c_3)\cdot \boldsymbol{A}, \tag{2.6}$$

$$c_5 = \frac{1}{\rho}(\nabla c_1 \times \nabla c_2)\cdot \nabla c_3 \tag{2.7}$$

are conserved in addition to c_1 and c_2 where $\boldsymbol{A} = \boldsymbol{v} - \nabla W$. He demonstrated that ρ, $\boldsymbol{A}$ and θ can be derived from $c_1, \ldots, c_5$ at least if degenerate situations are excluded. Thus $\boldsymbol{v}$ can be determined if W is known. Note that $\boldsymbol{q} = \nabla \times \boldsymbol{A}$ can be determined even if W is not known. If the set c_1–c_5 is to be complete, we must be able to eliminate (2.4). In other words, W must be derived from $c_i, i = 1-5$:

$$W = \mathscr{W}(c_i). \tag{2.8}$$

The operator $\mathscr{W}$ does not depend on time. Since (2.8) must hold at $t = 0$ as well as at any time $t > 0$ and since the initial distribution of all c_i can be chosen almost arbitrarily the condition $W = 0$ at $t = 0$ implies that the operator $\mathscr{W}$ yields $W = 0$ at any time, in contradiction to (2.4). Thus W is not completely determined through c_{1-5} and Hollmann's set is shown to be incomplete. So far we have essentially recovered Dikyi's (1972) result. We cannot exclude the possibility that one more invariant c_6 will be found. This new invariant is useful only if it does not depend on $c_1, \ldots, c_5$, i.e. there should be no relation $c_6 = \mathscr{C}_6(c_1, \ldots, c_5)$. Of course, we still can derive $\boldsymbol{A}$ from $c_1 \ldots, c_5$. To derive $\boldsymbol{v}$ we have to know W. This requires W to be expressed in terms of $c_1, \ldots, c_6$. Since $W = 0$ at $t = 0$ it follows $W = 0$ at any time. Thus c_1–c_6 is not a complete set. As a matter of fact c_6 is not independent of c_1, c_5. If $c_1, \ldots, c_5$ are specified at $t = 0$ we can determine the initial fields of velocity and density. Since $c_6 = c_6(\rho, \boldsymbol{v}, \theta)$ the initial distribution of c_6 can be derived from $c_1, \ldots, c_5$, i.e. c_6 depends on $c_1, \ldots, c_5$. There is still the possibility that a complete set exists for (2.1)–(2.3) but not for (2.1)–(2.3), (2.4). However, if that were the case, we need only add c_3 and c_4 to this set to obtain a complete set for (2.1)–(2.3), (2.4) in contradiction to what has been found above. It follows that a complete set of conservation laws does not exist for three-dimensional adiabatic flow. We can arrive at the same result by inserting $c_1, \ldots, c_5$ in (2.4)

$$\frac{\partial W}{\partial t} + (\boldsymbol{A} - \nabla W)\cdot \nabla W = \tfrac{1}{2}(\boldsymbol{A} - \nabla W)^2 - \phi - c_p(c_1(R(\nabla c_1 \times \nabla c_2)\cdot \nabla c_3/c_5\, p_0)^\kappa)^{1/(1-\kappa)}. \tag{2.9}$$

Since $\boldsymbol{A}(\boldsymbol{x}, t)$ can be derived from c_1–$c_5(\boldsymbol{x}, t)$ and $W(\boldsymbol{x}, 0) = 0$ it follows that W can be determined through (2.9) if $c_1, \ldots, c_5$ are known at all times $< t$. However it is not possible to determine W if all $c_1, \ldots, c_5$ are just known at time t as is suggested by (2.8).

3. Homogeneous fluids

3.1. *Shallow-water flow*

The equations of shallow-water flow are

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -g\nabla H, \tag{3.1}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -H\nabla\cdot \boldsymbol{v}, \tag{3.2}$$

where H is the depth of the fluid and $\boldsymbol{v} = (u, v)$. We shall show that potential vorticity

$$c_1 = q/H \quad \left(q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

is the only invariant of (3.1), (3.2) so that a complete set of conservation laws does not exist. We follow again Hollmann (1965) in our proof. Hollmann extended (3.1), (3.2) by adding an action function W governed by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \tfrac{1}{2}\boldsymbol{v}^2 - gH, \tag{3.3}$$

with $W = 0$ at $t = 0$. Then

$$c_2 = (\boldsymbol{k}\boldsymbol{A} \times \boldsymbol{\nabla} c_1)/H, \tag{3.4}$$

$$c_3 = J(c_1, c_2)/H \tag{3.5}$$

are invariants where again $\boldsymbol{A} = \boldsymbol{v} - \boldsymbol{\nabla} W$ and $\boldsymbol{v}(\boldsymbol{x}, t)$, $H(\boldsymbol{x}, t)$ can be determined if c_1–c_3, $W(\boldsymbol{x}, t)$ are known. Again it is straightforward to show that (2.8) cannot hold and that there is no complete set of invariants for (3.1), (3.2). In particular, W cannot be related diagnostically to invariants. Assume now that a new invariant c_4 is derived which is based on the original variables of shallow-water flow: $c_4 = c_4(\boldsymbol{v}, H)$. Let us integrate (3.3) and (1.1) with $i = 1, \ldots, 3$ in time starting from initial values $c_i(\boldsymbol{x}, 0)$, $i = 1, \ldots, 3$. Since $W = 0$ at $t = 0$ $\boldsymbol{v}(\boldsymbol{x}, 0)$, $H(\boldsymbol{x}, 0)$ can be derived from $c_i(\boldsymbol{x}, 0)$. This implies that $c_4(\boldsymbol{x}, 0)$ can be evaluated once the initial values $c_i(\boldsymbol{x}, 0)$ are known. Since all c_i are invariants, $c_4(\boldsymbol{x}, t)$ can be determined also at any time $t > 0$ from $c_i(\boldsymbol{x}, t)$, $i = 1, \ldots, 3$: $c_4 = \mathscr{C}_4(c_1, c_2, c_3)$. Moreover,

$$\boldsymbol{v} = \mathscr{A}(c_1, c_2, c_3) + \boldsymbol{\nabla} W, \tag{3.6}$$

$$H = J(c_1, c_2)/c_3, \tag{3.7}$$

where the symbol $\mathscr{A}$ denotes the operator needed to derive $\boldsymbol{A}$ from c_1, c_2, c_3. We insert (3.6), (3.7) in $c_4 = c_4(\boldsymbol{v}, H)$ to obtain

$$c_4 \equiv \mathscr{C}_4(c_1, c_2, c_3) = c_4(\mathscr{A}(c_1, c_2, c_3) + \boldsymbol{\nabla} W, J(c_1, c_2)/c_3). \tag{3.8}$$

Clearly c_4 must be independent of $\boldsymbol{\nabla} W$ so that $c_4 = c_4(q, H)$ since it is only by applying the operator $\boldsymbol{\nabla}\times$ to (3.6) that we can remove the term $\boldsymbol{\nabla} W$ in (3.8).

If c_4 is independent of c_1 continuity yields the new invariant

$$c_5 = J(c_1, c_4)/H. \tag{3.9}$$

Of course c_1, c_4, c_5 are independent. However, by inserting the expressions

$$H = J(c_1, c_4)/c_5, q = c_1 J(c_1, c_4)/c_5$$

in $c_4 = c_4(q, H)$ we find immediately that c_1, c_4, c_5 are dependent. It follows $c_4 = c_4(c_1)$. Potential vorticity is the only invariant of shallow-water flow. This result can be readily extended to hydrostatic compressible three-dimensional flow. Potential vorticity $(\boldsymbol{k} \cdot \nabla_\theta \times \boldsymbol{v})\,\partial\theta/\partial p$ is the only invariant in that case (except θ, of course).

As for one-dimensional shallow-water flow it is straightforward to show that the flow equations cannot be reduced to a prognostic equation of first order for X_1. So no complete set exists.

3.2. *Two-dimensional flow*

Conservation of vorticity

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 0, \tag{3.10}$$

$$q = \nabla^2\psi \tag{3.11}$$

allows the determination of the flow for all times since $\boldsymbol{v} = \boldsymbol{k} \times \nabla\psi$ can be derived from q. This statement is strictly correct only if there is no potential flow. Owing to our choice of periodic boundary conditions translation at constant speed is the only admissible type of potential flow. It can be removed by a simple transformation of coordinates. Thus (3.10) is essentially the complete set we are searching for.

3.3. *Three-dimensional homogeneous flow*

The basic equations are

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla p/\rho_0, \tag{3.12}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{3.13}$$

(ρ_0, constant), and we shall show that a complete set of conservation laws exists if we extend (3.12), (3.13) to include (1.3). Ertel's (1942) theorem reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q} \cdot \nabla\lambda) = 0 \tag{3.14}$$

for homogeneous flow where λ must be conserved. If we choose $\lambda = X_i$ we obtain three conservation laws (1.1) with

$$c_i = \boldsymbol{q} \cdot \nabla X_i \tag{3.15}$$

(see also Serrin 1959). A complete set is formed by (1.3), (3.15). If $X_i(\boldsymbol{x}, t)$ and $c_i(\boldsymbol{x}, t)$ are known at a certain time, one can derive $\boldsymbol{q}$ from c_i and $\boldsymbol{v}$ can be determined from $\boldsymbol{q}$ because of (3.13). As a matter of fact, one out of the three invariants (3.15) is redundant since vorticity is determined completely if two of its components are known. This result can be readily extended to incompressible flows since the equation of continuity provides us with the additional conservation law $\mathrm{d}\rho/\mathrm{d}t = 0$ in that case.

4. Concluding remarks

We have shown that complete sets of conservation laws exist for periodic incompressible flow in three dimensions if the Lagrangian tracer equations (1.3) are added to the system. A complete set does not exist for shallow-water flow nor for the most general case of compressible adiabatic three-dimensional flow. Potential vorticity is the only invariant of shallow-water flow. A corresponding search for complete sets of conservation laws can be conducted for nearly geostrophic flow (Salmon 1985) and intermediate flow models (e.g. McWilliams & Gent 1980). New problems come up if we admit more complicated boundaries. In that case potential flow need not be trivial. For the sake of brevity we abstain here from a discussion of such complications.

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REFERENCES

BRYAN, K. 1987 Potential vorticity in the models of the ocean circulation. *Q. J. R. Met. Soc.* **113**, 713–733.

DIKYI, I. 1972 A comment on the adiabatic-flow invariants suggested by Hollmann. *Izv. Atmos. Ocean. Phys.* **8**, 321–323 (English Transl. 182–183).

ERTEL, H. 1942 Ein neuer hydrodynamischer Wirbelsatz. *Met. Z.* **59**, 277–281.

ERTEL, H. & ROSSBY, C. G. 1949 Ein neuer Erhaltungssatz der Hydrodynamik. *Sitz.-Ber. Dtsch. Akad. Wiss. Berlin, Kl. f. Math. u. allg. Naturw. Nr.* 1.

FORTAK, H. 1956 Zur Frage allgemeiner hydrodynamischer Wirbelsätze. *Gerl. Beitr. Geophys.* **65**, 283–294.

HOLLMANN, G. 1964 Ein vollständiges System hydrodynamischer Erhaltungssätze. *Arch. Met. Geoph. Biokl.* A **14**, 1–13.

HOLLMANN, G. 1965 Über eine hydrodynamische Vertauschungsrelation und einige quasistatische Formen hydrodynamischer Erhaltungssätze. *Beitr. Phys. Atm.* **38**, 57–65.

HOSKINS, B., MCINTYRE, M. & ROBERTSON, A. 1985 On the use and significance of isentropic potential vorticity maps. *Q. J. R. Met. Soc.* **111**, 877–946.

MCWILLIAMS, J. & GENT, P. 1980 Intermediate models of planetary circulations in the atmosphere and the ocean. *J. Atmos. Sci.* **37**, 1657–1678.

MOBBS, S. 1981 Some vorticity theorems and conservation laws for non-barotropic fluids. *J. Fluid Mech.* **108**, 475–483.

SALMON, R. 1985 New equations for nearly geostrophic flow. *J. Fluid Mech.* **153**, 461–477.

SERRIN, J. 1959 Mathematical principles of classical fluid mechanics. In *Handbuch der Physik*, Vol. VIII/1; *Strömungsmechanik* (ed. Flügge und Truesdell), pp. 125–262. Springer.